ESTIMATING THE PARAMETERS OF PEARSON TYPE III POPULATIONS FROM SINGLY AND DOUBLY TRUNCATED OR CENSORED SAMPLES OF GROUPED OBSERVATIONS

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(Communicated by Dr. V. S. Huzurbazar, F.S.S., F.N.I.)

1. Introduction

Several examples, where truncated samples from Type III population arise, have been cited by Chapman (1956). The problem of estimating the parameters of the Pearson Type III population assuming various forms of truncation has been studied by Cohen (1950), Des Raj (1953), Chapman (1956) and others, mainly in the two cases: (*a*) to estimate the location parameter, μ, and the scale parameter, σ, when α_3, the third standard moment is known; and (*b*) to estimate μ, σ, and α_3 when only the form of the population is known. These authors in their treatment assume that the individual observations are available. In practice we often meet with grouped observations (*see* Chapman, 1956). Cohen (1950), and Chapman (1956) have dealt with grouped observations also. But, Cohen's estimates are inefficient (*see* Kenney and Keeping, 1951, pp. 376); and he has not evaluated the variances of his estimates. Though Chapman's estimates are efficient when individual observations are available, the same is not true when the observations are available only in groups; for in that case his equation (5) may not hold exactly. Moreover, Chapman's procedure can be efficiently used only for truncated ungrouped samples and not for censored samples, grouped or ungrouped. The purpose of the present paper is to present consistent and efficient estimates of the parameters of the Pearson Type III population from truncated and censored samples of grouped observations. The maximum likelihood procedure for truncated and censored samples of grouped observations, which the author (Swamy, 1960) used in the case of normal populations, is employed. As our procedure, under case (*b*), is too tedious to have much practical value only case (*a*) is considered in this paper.

We consider Cohen's forms of truncation, *viz.*, when the number of unmeasured observations is (1) unknown for each truncated tail, (2) known separately for each truncated tail, (3) known jointly for both truncated tails. The first case characterizes a doubly truncated sample and the last two a doubly censored sample.

The probability density function (P.D.F.) of a Pearson Type III population may be written as

$$f(x) = \frac{y_0}{\sigma}\left[1 + \frac{\alpha_3}{2}\left(\frac{x-\mu}{\sigma}\right)\right]^{(4/\alpha_3^2)-1} \cdot e^{-(2/\alpha_3)(x-\mu/\sigma)},$$

$$\mu - \frac{2\sigma}{\alpha_3} \leqslant x < \infty \tag{1}$$

where

$$y_0 = \left(\frac{4}{\alpha_3^2}\right)^{(4/\alpha_3^2)-1/2} \cdot e^{-(4/\alpha_3^2)}\left[\Gamma\left(\frac{4}{\alpha_3^2}\right)\right]^{-1}.$$

We consider independent random observations from the above population and assume that the observations with values equal to or between the points of truncation x_1 and x_{k+1} are alone measured. Let these measured observations be grouped into k specified classes with end-points $x_2, x_3, \cdots, x_{k+1}$.

Throughout the paper we adopt the following notation:

$$t_i = \frac{x_i - \mu}{\sigma} \qquad \text{for } i = 1, 2, \cdots, k+1,$$

$$\Phi(t) = y_0\left[1 + \frac{\alpha_3}{2}t\right]^{(4/\alpha_3^2)-1} \cdot e^{-(2/\alpha_3)t}, \qquad -\frac{2}{\alpha_3} \leqslant t < \infty.$$

$$G(t) = \int_{-(2/\alpha_3)}^{t} \Phi(t)\,dt.$$

For convenience we shall also use the abbreviations $G_i = G(t_i)$, $\Phi_i = \Phi(t_i)$ and $\Phi_i' = \Phi'(t_i)$ where dashes denote differentiation. Throughout the paper we shorten the summation symbol $\sum_{i=1}^{i=k}$ to merely Σ.

In § 3 we denote by $nI_{T_c}(j, l)$; $(j, l = \mu, \sigma)$ the Fisher's index of information about the parameters in the truncated sample of size n characterized by case (c), $c = 1, 2, 3$.

2. MAXIMUM LIKELIHOOD ESTIMATES OF THE PARAMETERS

Case (1): Doubly truncated samples: Number of unmeasured observations unknown for each truncated tail.—The density function is now written as

$$f(x) = \frac{y_0}{G_{k+1} - G_1} \cdot \frac{1}{\sigma}\left[1 + \frac{\alpha_3}{2}\left(\frac{x-\mu}{\sigma}\right)\right]^{(4/\alpha_3^2)-1}$$
$$\times e^{-(2/\alpha_3)(x-\mu/\sigma)}, \qquad x_1 \leqslant x \leqslant x_{k+1}. \tag{2}$$

Let the n measured observation with values in the sample range R be grouped by classes (x_i, x_{i+1}) $i = 1, 2, \cdots, k$. And let n_i be the number of observations falling in the i-th class, *i.e.*, between x_i and x_{i+1}. The probability P_{1i} of a sample observation falling in the i-th class is now

$$P_{1i} = \frac{y_0}{G_{k+1} - G_1} \int_{x_i}^{x_{i+1}} \left[1 + \frac{\alpha_3}{2}\left(\frac{x-\mu}{\sigma}\right)\right]^{(4/\alpha_3^2)-1}$$
$$\times e^{-(2/\alpha_3)(x-\mu/\sigma)} \cdot \frac{dx}{\sigma} \tag{3}$$
$$= \frac{G_{i+1} - G_i}{G_{k+1} - G_1}, \qquad i = 1, 2, \cdots, k.$$

The likelihood function of the sample is

$$L_1 = C_1 \prod_{i=1}^{k} P_{1i}^{n_i} = C_1 \prod_{i=1}^{k} \left(\frac{G_{i+1} - G_i}{G_{k+1} - G_1}\right)^{n_i}, \tag{4}$$

where C_1 is a constant.

Differentiating the logarithm of (4) and equating to zero, we obtain the maximum likelihood estimating equations

$$\frac{\partial \log L_1}{\partial \mu} = \sum \frac{n_i}{\sigma}\left[-\frac{\Phi_{i+1} - \Phi_i}{G_{i+1} - G_i} + \frac{\Phi_{k+1} - \Phi_1}{G_{k+1} - G_1}\right] = 0$$
$$\frac{\partial \log L_1}{\partial \sigma} = \sum \frac{n_i}{\sigma}\left[-\frac{t_{i+1}\Phi_{i+1} - t_i\Phi_i}{G_{i+1} - G_i} + \frac{t_{k+1}\Phi_{k+1} - t_1\Phi_1}{G_{k+1} - G_1}\right] = 0. \tag{5}$$

These equations may be written as

$$\Sigma n_i \left[Z_{11i} - Z_{11}(t_1, R)\right] = 0$$
$$\Sigma n_i \left[Z_{21i} - Z_{21}(t_1, R)\right] = 0 \tag{6}$$

where

$$Z_{pqi} = Z_{ppi}(t_i, y_i) = (-1)^q \frac{t_{i+1}^{p-1} \cdot \Phi_{i+1}^{q-1} - t_i^{p-1} \cdot \Phi_i^{q-1}}{G_{i+1} - G_i},$$

$(p, q = 1, 2, \text{etc.})$

y_i is the width of the i-th class in units of the standard deviation. The equations (6) are to be solved for getting the maximum likelihood estimates of μ and σ. They can be readily solved by using the tables of Z-functions. The Z-functions are extensively tabulated by the author with the aid of "Salvosa's Tables of Pearson's Type III Function" and are available in the Statistical Laboratory, University of Poona. Even in the absence of these tables of Z-functions the likelihood equations can be easily solved with the help of Salvosa's tables themselves. The method of solving the equations is illustrated by a numerical example in § 5.

Case (2) : *Doubly censored samples: Number of unmeasured observations known separately for each truncated tail.*—In this case a random sample of size $N = r + n + s$ is drawn from a population with the P.D.F. given by (1). It is known that r of the observations are in the left truncated tail and s of the observations are in the right truncated tail. Now n is the number of measured observations with values equal to or between the points of truncation. As before, these observations are grouped into k classes with end-points $x_2, x_3, \cdots, x_{k+1}$. Let the frequency in the i-th class be n_i, $(i = 1, 2, \cdots, k)$.

This time the probability P_{2i} of an observation falling in the i-th class is

$$P_{2i} = y_0 \int_{x_i}^{x_{i+1}} \left[1 + \frac{\alpha_3}{2}\left(\frac{x-\mu}{\sigma}\right)\right]^{(4/\alpha_3^2)-1} \times e^{(-2/\alpha_3)(x-\mu/\sigma)} \cdot \frac{dx}{\sigma}, \tag{7}$$

$$= G_{i+1} - G_i \qquad \text{for} \quad i = 1, 2, \ldots, k.$$

The likelihood function of the sample in this case may be written as

$$L_2 = C_2 \cdot G_1^r \cdot (1 - G_{k+1})^s \cdot \prod_{i=1}^{k} (G_{i+1} - G_i)^{n_i}, \tag{8}$$

where C_2 is a constant. Differentiating the logarithm of (8) and equating to zero we get the maximum likelihood estimating equations in terms of the Z-functions as

$$r \,.\, Z_{11}(t_1, \tau) + s \,.\, Z_{11}(t_{k+1}, \infty) + \Sigma n_i Z_{11i} = 0$$

$$r \,.\, Z_{21}(t_1, \tau) + s \,.\, Z_{21}(t_{k+1}, \infty) + \Sigma n_i Z_{21i} = 0 \tag{9}$$

where $\tau = -(2/\alpha_3 + t_1)$. The manner and method of solving these equations is exactly the same as in case (1).

Case (3) :—*Doubly censored samples: Number of unmeasured observations known jointly for the two truncated tails.*—This case is similar to the previous case (2) except that r and s are not known separately. The only information available is that out of the sample of size N, n of the observations are in the sample range R and the remaining $(N - n)$ observations are either in the left truncated tail or the right truncated tail. Now $N - n = r + s$. In this case the P.D.F. and the probability P_{3i} of an observation falling in the i-th class are the same as the previous ones, *i.e.*, case (2). But the likelihood function of the sample is

$$L_3 = C_2 \,.\, (1 - G_{k+1} + G_1)^{N-n} \,.\, \prod_{i=1}^{k} (G_{i+1} - G_i)^{n_i}. \tag{10}$$

Differentiating the logarithm of (10) and equating to zero we get the maximum likelihood estimating equations in terms of the Z-functions as

$$\Sigma n_i Z_{11i} - (N - n)\, Q_{11} = 0$$

$$\Sigma n_i Z_{21i} - (N - n)\, Q_{21} = 0 \tag{11}$$

where

$$Q_{pq} = (-1)^q \frac{t_{k+1}{}^{p-1} \,.\, \Phi_{k+1}{}^{q-1} - t_1{}^{p-1} \,.\, \Phi_1{}^{q-1}}{1 - G_{k+1} + G_1}$$

$(p,\ q = 1,\ 2,\ \text{etc.})$.

The manner of solving the equations is the same as in the previous cases.

Note that the Q-functions are functions of the truncation points and can be evaluated with the help of "Salvosa's Tables of Pearson's Type III Functions".

3. Precision of the Estimates*

The large sample (asymptotic) variance-covariance matrix of the estimates, in any case, is the reciprocal of the information matrix

$$\begin{Vmatrix} -E\left[\frac{\partial^2 \log L_c}{\partial \mu^2}\right] & -E\left[\frac{\partial^2 \log L_c}{\partial \mu \partial \sigma}\right] \\ -E\left[\frac{\partial^2 \log L_c}{\partial \mu \partial \sigma}\right] & -E\left[\frac{\partial^2 \log L_c}{\partial \sigma^2}\right] \end{Vmatrix}$$

$$(c = 1,\ 2,\ 3), \qquad (12)$$

where E stands for mathematical expectation.

Case (1).—In this case as

$$n \to \infty, \quad \frac{n_i}{n} \to P_{1i}$$

and hence the information indices are

$$-\underset{n\to\infty}{P\lim}\frac{\sigma^2}{n}\frac{\partial^2 \log L_1}{\partial \mu^2}$$

$$= \Sigma P_{1i}\,[Z_{11i}^2 - Z_{11}^2(t_1, R)] = I_{T_1}(\mu),$$

$$-\underset{n\to\infty}{P\lim}\frac{\sigma^2}{n}\frac{\partial^2 \log L_1}{\partial \mu \partial \sigma}$$

$$= \Sigma P_{1i}\,[Z_{11i}Z_{21i} - Z_{11}(t_1, R)\,.\,Z_{21}(t_1, R)] = I_{T_1}(\mu, \sigma),$$

and (13)

$$-\underset{n\to\infty}{P\lim}\frac{\sigma^2}{n}\frac{\partial^2 \log L_1}{\partial \sigma^2}$$

$$= \Sigma P_{1i}\,[Z_{21i}^2 - Z_{21}^2(t_1, R)] = I_{T_1}(\sigma).$$

The large sample variance-covariance matrix of the estimates is the reciprocal of

$$\frac{n}{\sigma^2}\,\|\,I_{T_1}(j, l)\,\|\,(j,\ l = \mu,\ \sigma).$$

Case (2).—This time as

$$N \to \infty, \quad \frac{n_i}{N} \to P_{2i}, \quad \frac{r}{N} \text{ and } \frac{s}{N}$$

stocastically converge to G_1 and $(1 - G_{k+1})$ respectively. Hence the information indices are

* The expressions for $I_{T_0}(j, l)$, $(j, l = \mu, \sigma)$ given in the author's earlier paper (Swamy, 1960) could have been further simplified, as is done in this paper.

$$-\underset{N\to\infty}{P\lim}\frac{\sigma^2}{N}\frac{\partial^2 \log L_2}{\partial\mu^2}$$
$$= G_1 \cdot Z_{11}{}^2(t_1, \tau) + (1 - G_{k+1}) \cdot Z_{11}{}^2(t_{k+1}, \infty)$$
$$+ \Sigma P_{2i}Z_{11i}{}^2 = I_{T_2}(\mu),$$

$$-\underset{N\to\infty}{P\lim}\frac{\sigma^2}{N}\frac{\partial^2 \log L_2}{\partial\mu\partial\sigma}$$
$$= G_1 \cdot Z_{11}(t_1, \tau)\, Z_{21}(t_1\, \tau)$$
$$+ (1 - G_{k+1})\, Z_{11}(t_{k+1}, \infty)\, Z_{21}(t_{k+1}, \infty)$$
$$+ \Sigma P_{2i}Z_{11i}Z_{21i} = I_{T_2}(\mu, \sigma), \tag{14}$$

and

$$-\underset{N\to\infty}{P\lim}\frac{\sigma^2}{N}\frac{\partial^2 \log L_2}{\partial\sigma^2}$$
$$= G_1 \cdot Z_{21}{}^2(t_1, \tau) + (1 - G_{k+1}) \cdot Z_{21}{}^2(t_{k+1}, \infty)$$
$$+ \Sigma P_{2i}Z_{21i}{}^2 = I_{T_2}(\sigma).$$

The large sample variance-covariance matrix of the estimates is the reciprocal of the information matrix

$$\frac{N}{\sigma^2}\| I_{T_2}(j, l) \|.$$

Case (3).—In this case as

$$N \to \infty, \quad \frac{n_i}{N} \to P_{3i}, \quad \frac{N-n}{N} \to (1 - G_{k+1} + G_1),$$

the degree of truncation and hence the information indices are given by

$$-\underset{N\to\infty}{P\lim}\frac{\sigma^2}{N}\frac{\partial^2 \log L_3}{\partial\mu^2}$$
$$= (1 - G_{k+1} + G_1) \cdot Q_{11}{}^2 + \Sigma P_{3i}Z^2_{11i} = I_{T_3}(\mu),$$

$$-\underset{N\to\infty}{P\lim}\frac{\sigma^2}{N}\frac{\partial^2 \log L_3}{\partial\mu\partial\sigma}$$
$$= (1 - G_{k+1} + G_1) \cdot Q_{11}Q_{21} + \Sigma P_{3i}Z_{11i}Z_{21i} = I_{T_3}(\mu, \sigma), \tag{15}$$

and

$$- \underset{N \to \infty}{P \lim} \frac{\sigma^2}{N} \frac{\partial^2 \log L_3}{\partial \sigma^2}$$

$$= (1 - G_{k+1} + G_1) \cdot Q_{21}^2 + \Sigma P_{3i} Z^2_{21i} = I_{T_3}(\sigma).$$

The large sample variance-covariance matrix of the estimates is obtained by taking the reciprocal of

$$\frac{N}{\sigma^2} \| I_{T_3}(j, l) \|.$$

4. Singly Truncated or Censored Samples

We shall now obtain the results for the singly truncated and censored samples as special cases of (1) and (2) above. It is obvious that under singly censored samples cases (2) and (3) above are identical. If only the left tail is truncated, $t_{k+1} \to \infty$ while, if the right tail alone is truncated, $-(2/\alpha_3) \leftarrow t_1$. We shall consider the case when the left tail alone is truncated. Now $t_{k+1} \to \infty$, $\Phi_{k+1} \to 0$, $t_{k+1} \cdot \Phi_{k+1} \to 0$ $G_{k+1} \to 1$ and $s = 0$.

Case (1): *Singly truncated samples*.—The maximum likelihood estimating equations in this case are

$$\Sigma n_i [Z_{11i} - Z_{11}(t_1, \infty)] = 0$$
$$\Sigma n_i [Z_{21i} - Z_{21}(t_1, \infty)] = 0. \tag{16}$$

and the information indices are

$$- \underset{n \to \infty}{P \lim} \frac{\sigma^2}{n} \frac{\partial^2 \log L_1}{\partial \mu^2}$$

$$= \Sigma P_{1i} [Z_{11i}^2 - Z_{11}^2(t_1, \infty)] = I_{T_1}(\mu),$$

$$- \underset{n \to \infty}{P \lim} \frac{\partial^2}{n} \frac{\partial^2 \log L_1}{\partial \mu \partial \sigma}$$

$$= \Sigma P_{1i} [Z_{11i} Z_{21i} - Z_{11}(t_1, \infty) \cdot Z_{21}(t_1, \infty)] = I_{T_1}(\mu, \sigma) \tag{17}$$

and

$$- \underset{n \to \infty}{P \lim} \frac{\sigma^2}{n} \frac{\partial^2 \log L_1}{\partial \sigma^2}$$

$$= \Sigma P_{1i} [Z^2_{21i} - Z_{21}^2(t_1, \infty)] = I_{T_1}(\sigma),$$

where now

$$P_{1i} = \frac{G_{i+1} - G_i}{1 - G_1}.$$

The large sample variance-covariance matrix of the estimates is given by

$$\frac{\sigma^2}{n} \| I_{T_1}(j, l) \|^{-1}.$$

Case (2): *Singly censored samples.*—In this case the maximum likelihood estimating equations are

$$r \cdot Z_{11}(t_1, \tau) + \Sigma n_i Z_{11i} = 0$$

$$r \cdot Z_{21}(t_1, \tau) + \Sigma n_i Z_{21i} = 0. \tag{18}$$

The information indices are

$$-\underset{N \to \infty}{P \lim} \frac{\sigma^2}{N} \frac{\partial^2 \log L_2}{\partial \mu^2}$$

$$= G_1 \cdot Z_{11}{}^2(t_1, \tau) + \Sigma P_{2i} Z_{11i}{}^2 = I_{T_2}(\mu),$$

$$-\underset{N \to \infty}{P \lim} \frac{\sigma^2}{N} \frac{\partial^2 \log L_2}{\partial \mu \partial \sigma}$$

$$= G_1 \cdot Z_{11}(t_1, \tau) Z_{21}(t_1, \tau) + \Sigma P_{2i} Z_{11i} Z_{21i} = I_{T_2}(\mu, \sigma),$$

and (19)

$$-\underset{N \to \infty}{P \lim} \frac{\sigma^2}{N} \frac{\partial^2 \log L_2}{\partial \sigma^2}$$

$$= G_1 \cdot Z_{21}{}^2(t_1, \tau) + \Sigma P_{2i} Z_{21i}{}^2 = I_{T_2}(\sigma).$$

The large sample variance-covariance matrix of the estimates is given by

$$\frac{\sigma^2}{N} \| I_{T_2}(j, l) \|^{-1}.$$

Note that the $I_{T_e}(j, l)$'s in the singly truncated and censored cases are not the same as the corresponding $I_{T_e}(j, l)$'s in the doubly truncated and censored cases. It may also be noted that the procedure of this paper can also be used, without loss of much information, for ungrouped observations by grouping the individual sample observations into equidistant classes.

5. A Numerical Example

The practical application of results of this paper, and the iterative method employed in solving the maximum likelihood estimating equations can be best illustrated by considering the following numerical example of Cohen (1950).* Cohen's data consist of a grouped sample distribution of the weights of 629 University of Washington Freshmen. The class interval of this distribution is 11 lb., and there are 11 classes in the complete sample. It is known that the sample is from a Type III population with $\alpha_3 = \cdot 6$. For our present illustration, a doubly truncated sample was obtained from Cohen's data by truncating his complete sample on the left at 110·5 lb., and on the right at 187·5 lb. The truncated sample consists of seven classes. The frequencies in these seven classes are as follows:

Class Boundaries ..	111–	122–	133–	144–	155–	166–	177–
Frequency	43	138	162	129	82	35	16

In the following we shall obtain the maximum likelihood estimates of μ and σ of the population from which the sample is drawn, and their large sample variance-covariance matrix in the case of doubly truncated samples. The estimates and their large sample variances in the other cases can be similarly obtained. Throughout only simple linear interpolation is employed.

Case (1): *Doubly Truncated Samples.*—In general, the first and the second moments about the origin of the truncated sample can be taken as the first working approximation to $\hat{\mu}$ and $\hat{\sigma}$. In the present example, as the first and the second moments about the origin of the truncated sample are 142·327 and 17·127 respectively; a working approximation to $\hat{\sigma}$ is taken as 18. With $\hat{\sigma} = 18$ [estimates are indicated by circumflex (^)] and $(144 - \hat{\mu})/\hat{\sigma} = \cdot 2$ respectively 0 we obtain the table given in the following page.

By simple linear interpolation we see that with $\hat{\sigma} = 18$, $\Sigma n_i [Z_{11i} - Z_{11}(t_1, R)] = 0$ for $\hat{\mu} = 142\cdot 7778$. With this value of $\hat{\mu}$, using Z_{21} function we have $\Sigma n_i [Z_{21i} - Z_{21}(t_1, R)] = -19\cdot 0706$ for $\hat{\sigma} = 17$ and $\Sigma n_i [Z_{21i} - Z_{21}(t_1, R)] = 6\cdot 4262$ for $\hat{\sigma} = 18$. Hence, with $\hat{\mu} = 142\cdot 7778$, $\Sigma n_i [Z_{21i} - Z_{21}(t_1, R)] = 0$ for $\hat{\sigma} = 17\cdot 748$. Thus, the first cycle of the iterative process gives the estimates of μ and σ

* The author is thankful to Dr. A. C. Cohen, Jr., for furnishing him with the class boundaries and the frequencies for the complete numerical example employed in his paper (Cohen, 1950).

x_i	$\frac{x_i-\hat{\mu}}{\hat{\sigma}}$	n_i	$Z_{11i}-Z_{11}$ (t_1, R)	$\frac{x_i-\hat{\mu}}{\hat{\sigma}}$	n_i	$Z_{11i}-Z_{11}$ (t_1, R)
111–	−1·633	43	−1·6993	−1·833	43	−2·1374
122–	−1·022	138	−1·0285	−1·222	138	− ·8542
133–	− ·411	162	·1026	− ·612	162	− ·0440
144–	·200	129	·5962	0	129	·5088
155–	·811	82	·9577	·612	82	·9073
166–	1·422	35	1·2336	1·222	35	1·2070
177–	2·033	16	1·4526	1·833	16	1·4418
	Total ..	605	$\Sigma n_i [Z_{11i}-Z_{11} (t_1, R)] = 23·4771$	Total ..	605	$\Sigma n_i [Z_{11i}-Z_{11} (t_1, R)] = -11·5682$

from doubly truncated sample as 142·7778 and 17·748 respectively. To ensure whether the estimates are better, the iterative process may be repeated with these values of $\hat{\mu}$ and $\hat{\sigma}$. However, in the present example, as the first approximations to $\hat{\mu}$ and $\hat{\sigma}$ are very near to the estimates of μ and σ obtained by Cohen (1950) both in the untruncated case and the singly truncated case, further approximation to $\hat{\mu}$ and $\hat{\sigma}$ are not obtained. Cohen's estimates of μ and σ when only the left tail is truncated are 142·74 and 17·43 respectively; the corresponding values from the complete sample are 142·25 and 17·59.

Taking $\hat{\mu} = 142·7778$ and $\hat{\sigma} = 17·748$ an estimate of the variance-covariance matrix of the above estimates is obtained as

$$·5206 \begin{Vmatrix} ·8503 & -·2332 \\ -·2332 & ·9122 \end{Vmatrix}^{-1}.$$

Hence $V(\hat{\mu}) = ·6582$ and $V(\hat{\sigma}) = ·6138$. These variances are expected to be smaller than the variances of the estimates of Cohen's (1950) paper; however, a ready comparison cannot be made as the variances of Cohen's estimates are not easily available.

The author hopes to investigate in a subsequent paper whether the difficulty involved in considering case (*b*) by the present method might be overcome by adopting Hartley's (1958) technique of dealing with incomplete data.

6. SUMMARY

The maximum likelihood estimating equations are derived to estimate the parameters of a Pearson Type III complete population from doubly truncated and censored random samples of grouped observations with known truncation points. Asymptotic variance-covariance matrices of these estimates are evaluated. The results for singly truncated and censored samples are obtained as special cases. Practical application of the results has been illustrated by a numerical example.

7. ACKNOWLEDGEMENT

The author wishes to acknowledge with gratitude Dr. V. S. Huzurbazar's encouragement and constant guidance in carrying through the present work. He likes to express his thanks to Professor K. V. Krishna Sastry and Dr. V. P. Bhapkar for their many helpful suggestions. He also wishes to thank the Government of India for awarding him a Research Training Scholarship.

8. REFERENCES

1. Cohen, A. C. Jr. .. "Estimating parameters of Pearson Type III population from truncated samples," *Journal of the American Statistical Associations*, 1950, **45**, 411–23.
2. Des Raj .. "Estimation of the parameters of Type III populations from truncated samples," *Ibid.*, 1953, **48**, 336–49.
3. Douglas, G. Chapman .. "Estimating the parameters of a truncated gamma distribution," *Ann. Math. Stat.*, 1956, **27**, 498–506.
4. Gjeddeback, N. F. .. "Contribution to the study of grouped observations. Application of the method of maximum likelihood in the case of normally distributed observations," *Skandinavisk Aktuarietidskrift*, 1949, **35**, 135–59.
5. Hartley, H. O. .. "Maximum likelihood estimation from incomplete data," *Biometrics*, 1958, **14**, 174–94.
6. Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics*, Part II, Van Nostrand, Inc., New York, 1951.
7. Salvosa, L. R. .. "Tables of Pearson's Type III functions," *Ann. Math. Stat.*, 1930, **1**, appended.
8. Swamy, P. S. .. "Estimating the mean and variance of a normal distribution from singly and doubly truncated samples of grouped observations," *Cal. Stat. Assoc. Bull.*, 1960, **9**, 145–56.